Efficient Incremental Penetration Depth Estimation between Convex Geometries Supplemental Document

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I. Derivation of the SQP Procedure

As explained in Section III.A of the original paper, we investigate the following optimization problem:

$$\begin{aligned} &\min_{v \in R^n} \quad ||v||_2 \\ &\text{subject to:} \quad v \in \text{boundary}(\mathscr{E}) \end{aligned} \tag{1}$$

where $\mathscr{E}$ is also a closed convex set in R^n that contains the origin. Different from Sec. III.C of the paper, we assume an explicit representation of $\mathscr{E}$ is available, such that we can compute a supporting hyperplane for each point $v \in \text{boundary}(\mathscr{E})$. As a result, this PD algorithm is "conceptual" and will be instantiated in Sec. III.C of the original paper.

The domain for the decision variable in Problem (1) only includes the boundary of $\mathscr{E}$, as shown in Fig. 1 (a). As the convex set $\mathscr{E}$ contains the origin, we can enlarge the input domain to everywhere except the inner of $\mathscr{E}$ and rewrite the optimization as

$$\begin{aligned} &\min_{v \in R^n} \quad ||v||_2 \\ &\text{subject to:} \quad v \in (R^n \setminus \text{inner}(\mathscr{E})) \end{aligned} \tag{2}$$

where $R^n \setminus \text{inner}(\mathscr{E})$ is the complement of $\text{inner}(\mathscr{E})$, an illustration if shown in Fig 1 (b).

We would like to solve Problem 2 using a modified SQP procedure [2], [1]. SQP operates by alternating between locally convexifying the costs/constraints and solving the QP sub-problem. For Problem 2, the cost function is the L2-norm of the decision variable v, which is convex and quadratic. Thus, we only need to linearize the non-convex inequality constraint in Problem 2.

Suppose for some SQP iteration k, the point v_k is on the boundary of $\mathscr{E}$. The local linearization of the inequality constraint at v_k is actually a half-space separated by the supporting hyperplane at v_k. An illustration is shown in Fig 1 (c). Obviously, the half-space does not contain the origin. Moreover, the QP sub-problem becomes finding the minimum distance point in the half-space to the origin. Usually, QP sub-problems in a SQP procedure require a trust region to ensure decreasing cost. This is unnecessary for our setup as Problem 2 is a different-of-convex problem. Let y_k be the projection of origin onto the half-space, which is also the optimal solution to the QP sub-problem.

For a point y_k that is not on the boundary of the $\mathscr{E}$, the original SQP procedure require a similar local linearization of the inequality constraint at y_k and subsequent QP sub-problem. This linearization is also a half space whose separating plane passes through y_k. We modify the SQP procedure to replace QP sub-problem by projecting the y_k back to the boundary of $\mathscr{E}$ at v_{k+1}, and the overall PD algorithm is shown in Algorithm 1. We use this modification because Algorithm 1 is a conceptual algorithm that will be instantiated using MPR subroutine in Section III.B of the original paper. It is easy to prove the minimum penetration distance estimated by Algorithm 1 converges to a local optimal solution: for each iteration we have $|v_{k+1}| \leq |y_k| \leq |v_k|$ and the estimated minimum distance does not increase.

Algorithm 1 SQP for Problem. 2

Require: $\mathscr{E}$ that supports compute_supporting_hyperplane($\cdot$)
Require: $v_{\text{init}} \in \text{boundary}(\mathscr{E})$
  $v_0 \leftarrow v_{\text{init}}$
  while $k = 0, 1, 2, \ldots$ **do**
    $n_{\text{plane\_}k} \leftarrow$ compute_supporting_hyperplane($\mathscr{E}$, v_k)
    $\triangleright$ Plane defined by normal $n_{\text{plane\_}k}$ and a point v_k on it
    $y_k \leftarrow$ project_point_to_plane(O, Plane(v_k, $n_{\text{plane\_}k}$))
    $v_{k+1} =$ boundary_intersection(O_to_y_k, boundary($\mathscr{E}$))
  end while
  return v_k

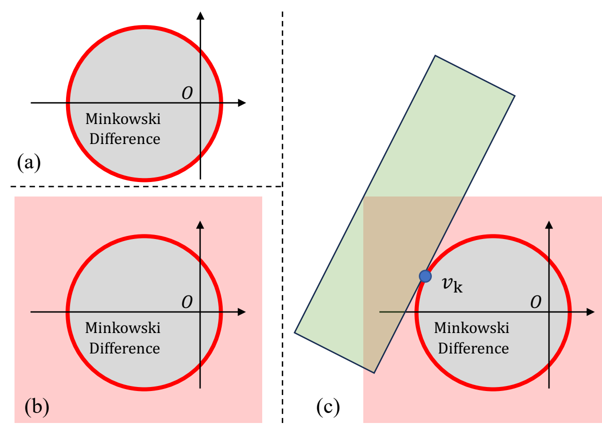

Fig. 1: Formulation of the SQP problem. (a) shows the original optimization in Problem 1 whose domain is the boundary; (b) shows the formulation in Problem 2 with extended domain; (c) shows the convex relaxation of the optimization problem at a point v_k.

II. Convergence Proof of the Proposed Algorithm

Build upon Algorithm 1, we propose a new PD algorithm in Section III.C of the original paper by introducing a MPR subroutine instantiation and a shortcut mechanism, as shown in the Algorithm 4 of the original paper. The new PD algorithm use penetration direction d as the decision variable, and $d = \text{normalized}(v)$. The penetration point v is computed using the MPR subroutine.

The convergence properties of the new PD algorithm in Section III.C of the original paper is almost the same as the Algorithm 1: in each iteration k that we update the direction d_k to d_{k+1}, the corresponded penetration point v_k satisfies $|v_{k+1}| \leq |v_k|$ following the shortcut mechanism in Section III.C. As a result, the minimum penetration distance estimated by our PD algorithm converges to a local optimal solution.

REFERENCES

[1] S. Boyd, S. P. Boyd, and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[2] P. E. Gill, W. Murray, and M. A. Saunders. Snopt: An sqp algorithm for large-scale constrained optimization. *SIAM review*, 47(1):99–131, 2005.